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INFINITESIMAL PROPERTIES OF LINES IN S_4 *WITH APPLICATIONS TO CIRCLES IN* S_3.

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Introduction.

In a recent paper (to be published in the American Journal, 1911) the author discussed properties of systems of lines in S_4 involving differentials of the first order, that is, simple tangency, and interpreted the results in the geometry of the circle in space of three dimensions. It is the purpose of the present paper to discuss properties of lines and circles involving second and higher differentials. The methods of the first paper are not applicable to this case. Here lines are represented by points of a six-dimensional spread which stands in a space of nine dimensions. The problem is, then, simply one of differential geometry in higher space.

1. *The coordinates.*[1] In S_4 let the line be defined by two points (homogeneous coordinates). Then the coordinates of a line are the two row determinants of the matrix,

$$\left\| \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{matrix} \right\|.$$

Adopting the usual notation

$$p_{ik} = x_i y_k - x_k y_i,$$

we see that there are ten coordinates, since

$$p_{ik} = -p_{ki}, \qquad p_{ii} = 0.$$

These ten coordinates are connected by the five quadratic relations

$$\Omega_i \equiv p_{kl}p_{mn} + p_{km}p_{nl} + p_{kn}p_{lm} = 0 \tag{1}$$

[1] Castelnuovo, "Ricerche di geometria della retta nello spazio a quattro dimensione," Atti del Reale Istituto Veneto, series 7, **2**, 1890–91. In this paper only linear systems of lines have been considered.

($i = 1, 2, 3, 4, 5$, and $k\ l\ m\ n$ is a permutation of the remaining numbers after i is chosen). It is well known, however, that of these five relations Ω_i only three are distinct.

Now, letting p_{ik} be the ten homogeneous coordinates of a point in a space of nine dimensions S_9, the points which represent lines in S_4 are the points common to the five quadrics

$$\Omega_i = 0.$$

The intersection of these quadrics is known to be of order five and dimensions six. In our discussions, then, we shall be interested only in the points of this variety which we shall indicate by Φ. The following properties can be easily verified.

(a) The hyperplanes

$$\sum_1^5 i\, \xi_i p_{ik} = 0, \qquad k = 1, 2, 3, 4, 5, \tag{2}$$

cut Φ in quadrics V_4^2 of order two and dimensions four; hence Φ contains ∞^4 such quadrics.

(b) The S_3 defined by

$$p_{ik} = l_1 p_{ik}' + l_2 p_{ik}'' + l_3 p_{ik}''' + l_4 p_{ik}'''',$$

where p', p'', p''', p'''' correspond to lines passing through a common point, lies in Φ.

(c) The S_2 defined by

$$p_{ik} = l_1 p_{ik}' + l_2 p_{ik}'' + l_3 p_{ik}''',$$

where p', p'', p''' represent lines which lie in the same plane, also lies in Φ. It is easily seen that the planes (c) and the spaces (b) do not in general intersect. It can also be easily shown that two quadrics (a) intersect in a plane (c).

A pencil of lines in S_4 is represented by a line which is contained in Φ. The points of a quadric (a) represent the lines of S_4 which lie in a space of three dimensions S_3. A space (b) represents the lines in S_4 which pass through a fixed point, and a plane (c) represents the lines which lie in a fixed plane.

The lines which cut a fixed line p_{ik}' are represented by the intersection of Φ with its tangent S_6 at the point p_{ik}'. (Here p_{ik}' is used to denote a line in S_4 or a point on Φ.)

$$\sum \frac{\partial \Omega_i}{p_{kl}} p_{kl}' = 0. \tag{3}$$

The intersection is of order three and dimensions four. The lines which cut two fixed lines will be represented by the intersection of two such tangent spaces S_6 with Φ; this is an ordinary quadric contained in Φ.

2. *Directions in ruled space of four dimensions.* In ruled space of four dimensions as in three dimensions a direction through a given line of a system is defined as the Chasles correlation which a ruled surface belonging to the system and passing through the given line determines; that is to say, a direction through a given line is determined by the given line and a line infinitely near to it. In the S_9 defined by the coordinates p_{ik} this corresponds to the tangent lines to the curves traced on the variety Φ, but since the points of S_9 which lie outside of Φ have no significance, the tangent lines in general will have none. We can, however, consider the polar of such a line. This polar will intersect Φ in an ordinary quadric cone, and since it can be looked upon as the intersection of two tangent spaces S_6 to Φ at infinitely near points of the given line, the points of this cone will represent lines of S_4 which cut two infinitely near lines, that is, will represent the special congruence determined by the two infinitely near lines. This congruence represents the Chasles correlation determined by the two lines. If the tangent line to Φ lies entirely in Φ, the quadratic cone and consequently the congruence will degrade. A curve all of whose tangents lie in Φ will represent a developable in S_4 because all the Chasles correlations are degenerate.

We saw that a tangent S_6 cuts Φ in a V_4^3. Then, if this S_6 is cut by an arbitrary hyperplane, the V_4^3 will be cut in a variety ϕ_3^3 of order three and dimensions three. Each point of Σ, the intersection of S_6 and the hyperplanes, will represent a direction through the line r corresponding to the point of tangency. The points of ϕ_3^3 will correspond to the special directions through r, that is, to the degenerate projectivities. The variety ϕ_3^3 and the space Σ in which it stands can be used instead of the ones used by the author in the paper previously referred to, and all the results there obtained can be obtained here. Properties of lines involving differentials of the first order are disposed of in this manner.

I. Properties of Five-Parameter Families of Lines.

3. In the following, for the convenience of writing, the coordinates $x_1, x_2 \ldots x_{10}$ will be used instead of $p_{12}, p_{13} \ldots p_{45}$. Also a variety on Φ and the system of lines which it represents will be denoted by the same symbols where there can be no ambiguity. The symbols $C_1, C_2 \ldots C_5$ will be used to denote linear systems of lines.

Consider a variety V_5 traced on Φ; that is, let the coordinates x be functions of the five parameters $u_1, u_2 \ldots u_5$. Here and following all the functions used are supposed to be continuous and to possess all partial derivatives of the n^{th} order. The tangent lines to V_5 will generate an S_6 determined by the six points [2]

$$(\pi_5) \qquad (x), \quad \left(\frac{\partial x}{\partial u_1}\right), \quad \left(\frac{\partial x}{\partial u_2}\right), \quad \left(\frac{\partial x}{\partial u_3}\right), \quad \left(\frac{\partial x}{\partial u_4}\right), \quad \left(\frac{\partial x}{\partial u_5}\right).$$

Through this S_5 will pass ∞^4 hyperplanes which are tangent likewise to V_5 (an hyperplane is said to be tangent to a variety if it contains the tangent space to the variety). Hence, *In each line of a five-parameter family V_5 there are ∞^4 C_5's tangent to it.*

The osculating planes of all the curves traced on V_5 which have the same tangent line in x generate an S_6 which contains the space π_5 tangent to V_5, as may be seen as follows :

Using Segre's [3] notation, the osculating planes will be determined by the three points

$$f = 0, \tag{4}$$

$$\Sigma f_i du_i = 0, \tag{5}$$

$$\Sigma f_{ik} du_i du_k + \Sigma f_i d^2 u_i = 0. \tag{6}$$

Tangent lines are lines which join (4) to any point of the space determined by f_1, f_2, f_3, f_4, f_5 and hence generate π_5. Now, if the du_i are held fixed (that is, the direction of the tangent is fixed) and the d^2u_i are allowed to vary, we see that the osculating plane will always stand in the S_6 defined by

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[2] A tangent line joins the points x and $x + dx$, or expanding x and $x + \Sigma \frac{\partial x}{\partial u_i} du_i$. As the du_i determine the direction along V_5, the space generated by these lines will be the S_5 named.

[3] "Su una classa di superficie ecc." Atti di Torino, 1907. Points are represented by their equations in hyperplanar coordinates or also with the first member of these equations. Thus

$$f = \Sigma \xi^{(i)} x^{(i)},$$

where $\xi^{(i)}$ are the hyperplanar coordinates. The point which describes a variety shall be represented by f. We may also speak of the point x, represented by f, or $\Sigma \xi^{(i)} x^{(i)} = 0$. Subscripts are reserved to denote derivatives; thus,

$$f_{ijk} = \frac{\partial^3 f}{\partial u_i \partial u_j \partial u_k}.$$

$$(7) \qquad f = 0,\ f_1 = 0,\ f_2 = 0,\ f_3 = 0,\ f_4 = 0,\ f_5 = 0,$$
$$\Sigma f_{ik} du_i du_k = 0.$$

Since there are ∞^4 planes obtained allowing d^2u_i to vary, this S_6 must be the locus. From (7) it is evident that the S_6 contains the tangent space π_5, which is determined by

$$f = 0,\ f_1 = 0 \ . \ . \ . \ f_5 = 0.$$

We will call the space defined by (7) an *hyperosculating* S_6. Now, if an S_6 containing π_5 is given, then a direction (values of du_i) can be determined so that the S_6 will hyperosculate V_5 along this direction. It is only necessary for the du_i to satisfy the relations

$$(8) \qquad \begin{aligned} (\xi,\ \Sigma f_{ik} du_i du_k) &= 0, \\ (\xi',\ \Sigma f_{ik} du_i du_k) &= 0, \\ (\xi'',\ \Sigma f_{ik} du_i du_k) &= 0, \end{aligned}$$

where the symbol $(\xi, x) = 0$ is used for $\Sigma\, \xi^{(i)} x^{(i)} = 0$ and the S_6 is defined as the intersection of the three hyperplanes

$$(\xi, x) = 0, \quad (\xi', x) = 0, \quad (\xi'', x) = 0.$$

Then the tangents to V_5 along which the S_6 is hyperosculating are determined by the values of du_i which satisfy (8). There are ∞^1 values of du_i which satisfy (8), and the tangents so defined generate a cone of order eight. Therefore *an S_6 passing through π_5 will hyperosculate V_5 along ∞^1 directions forming a cone of order eight.* This shows that the hyperosculating S_6 cuts V_5 in a surface which has a conical point of order eight. It is also evident that a hyperplane passing through a hyperosculating S_6 will cut V_5 in a V_4 which has a conical point of order two, since only the first of equations (8) would have to be satisfied. Similarly, an S_7 which contains the hyperosculating S_6 will cut V_5 in a V_3 which has a conical point of order four, since only two of equations (8) would have to be satisfied.

4. We have just seen that a hyperplane passing through π_5 cuts V_5 in a V_4 with a conical point in x, the point of tangency. Let us now examine how this cone varies as the hyperplane varies. The coordinates of the hyperplane will be

$$\xi_i = \lambda a_i + \mu \beta_i + \nu \gamma_i + \eta \delta_i$$

(α, β, γ, δ are the coordinates of four hyperplanes passing through π_5). The cones therefore will form a linear series of three parameters and hence will have sixteen generators in common. *These are the sixteen directions upon which rest many of the infinitesimal properties of higher order of five-parameter families of lines.*

The tangent space π_5 cuts V_5 in a curve which has a multiple point of order sixteen in x. The tangents at the multiple point are the sixteen directions noted above.

These sixteen directions correspond to the asymptotic directions on a surface in S_3 both in the sense that two successive tangent spaces π_5 intersect in a line joining the points of contact and in the sense that tangents to V_5 in one of these directions have three-point contact. The latter property can be seen by deriving the condition that a tangent line have three-point contact. From equations (4), (5), (6) we see that this would simply lead to the condition that the point

$$\Sigma f_{ik} du_i du_k = 0$$

should lie in the tangent π_5, because in that case the quantities d^2u_i could be so chosen that the three-point (4,) (5,) (6), that is, (x), (dx,) (d^2x), would lie on a line. The condition is, then, the vanishing of the determinants of the matrix

$$\left\| x \quad \frac{\partial x}{\partial u_1} \quad \frac{\partial x}{\partial u_2} \quad \frac{\partial x}{\partial u_3} \quad \frac{\partial x}{\partial u_4} \quad \frac{\partial x}{\partial u_5} \quad \Sigma \frac{\partial x}{\partial u_i \partial u_k} du_i du_k \right\|$$

and this is exactly the condition that defines the sixteen directions above.

5. In order to see that the tangent spaces at two consecutive points of V_5 along one of these sixteen directions intersect along the line joining the points of contact, we will first examine what corresponds to conjugate directions on V_5.[4] If we consider the points of V_5 as determined by the hyperplanes passing through them, then any point will be given by

$$(\xi, x) = 0,$$

where x is a function of the five u's. Then, if the tangent π_5 be looked upon as the intersection of four hyperplanes, the condition that the hyperplanes pass through x is

$$(9) \qquad (\xi', x) = 0, \quad (\xi'', x) = 0, \quad (\xi''', x) = 0, \quad (\xi^{\text{IV}}, x) = 0,$$

[4] The argument here is the same as that used by Segre for discussing a V_3 in S_5 in his 1907–8 lectures on "Geometria della retta."

and if this S_5 is tangent to V_5,

$$(10)\qquad \left(\xi', \frac{\partial x}{\partial u_1}\right) = 0, \quad \left(\xi', \frac{\partial x}{\partial u_2}\right) = 0, \quad \left(\xi', \frac{\partial x}{\partial u_3}\right) = 0,$$
$$\left(\xi', \frac{\partial x}{\partial u_4}\right) = 0, \quad \left(\xi, \frac{\partial x}{\partial u_5}\right) = 0,$$

and a similar set of equations for the hyperplanes ξ'', ξ''', ξ^{IV}. The S_5 infinitely near is determined by

$$(11)\qquad (\delta\xi', x) = 0, \ (\delta\xi'', x) = 0, \ (\delta\xi''', x) = 0, \ (\delta\xi^{\text{IV}}, x) = 0.$$

These two spaces will intersect in a line. If the point dx infinitely near to x is to lie in (11),

$$(12)\qquad (\delta\xi', dx) = 0 \ \ldots \ (\delta\xi^{\text{IV}}, dx) = 0.$$

Now, using δu_i to denote the direction of the line of intersection of (10) and (11), and du_i to denote the direction of the line joining x and $x + dx$, equations (12) take the form

$$(13)\qquad \left(\sum \frac{\partial \xi'}{\partial u_i}\delta u_i, \quad \sum \frac{\partial x}{\partial u_i} du_i\right) = \sum\left(\frac{\partial \xi'}{\partial u_i}, \frac{\partial x}{\partial u_j}\right)\delta u_i du_j = 0,$$

and similar equations for the other ξ's.

Differentiating (9) and (10) partially with respect to the u's, we have

$$\left(\frac{\partial \xi'}{\partial u_i}, \frac{\partial x}{\partial u_j}\right) + \left(\xi', \frac{\partial^2 x}{\partial u_i \partial u_j}\right) = 0.$$

Substituting these values in (13),

$$(14)\qquad \sum\left(\xi', \frac{\partial^2 x}{\partial u_i \partial u_j}\right)\delta u_i du_j = \left(\xi', \sum \frac{\partial^2 x}{\partial u_i \partial u_j}\right) = 0.$$

Now, if we put $du_i = \delta u_i$ we have the system of equations (8); hence that is the condition that should be verified if the two consecutive tangent spaces are to intersect in the line joining the points of contact.

Since V_5 is contained in Φ (the intersection of the five quadrics Ω_i), the tangents to V_5 which have three-point contact must lie entirely in Φ. The sixteen lines discussed above then represent pencils of lines in S_4 and may be called *osculating* pencils, since they contain three lines of V_5 infinitely near. Further the planes of these pencils are inflexional for

the cones of the system V_5 passing through the vertices of the pencils. Then

Through a line p of V_5 pass sixteen cones of V_5 which have inflexional planes, and sixteen curves of V_5 lying in planes passing through p have the line p for inflexional tangent.

The tangent π_5 to V_5 at x cuts V_5 in a curve which has in x a multiple point of order sixteen; hence *The ∞^4 linear series C_5 tangent V_5 in a line p cut V_5 in a ruled surface having p for multiple generator of order sixteen.*

6. The locus of all osculating S_4's of the curves traced on V_5 and passing through x and having the same osculating S_3 is a hyperplane S_8. To show this let us consider the S_4 as defined by the points

$$(x),\ (dx),\ (d^2x),\ (d^3x),\ (d^4x),$$

which in hyperplanar coordinates are

$$f = 0, \tag{4}$$

$$\Sigma f_i du_i = 0, \tag{5}$$

$$\Sigma f_{ij} du_i du_j + \Sigma f_i d^2u_i = 0, \tag{6}$$

$$\Sigma f_{ijk} du_i du_j du_k + 3\,\Sigma f_{ij} d^2u_i du_j + \Sigma f_i d^3u_i = 0, \tag{15}$$

$$\Sigma f_{ijkl} du_i du_j du_k du_l + 6\,\Sigma f_{ijk} du_i du_j d^2u_k + 3\,\Sigma f_{ij} d^2u_i d^2u_k + 4\,\Sigma f_{ij} d^3u_i du_j + \Sigma f_i d^4u_i = 0. \tag{16}$$

If d^4u_i are allowed to vary and all the other differentials are held fixed by reasoning similar to that previously used, it will be seen that the osculating S_4 will generate an S_8 determined by the nine points

$$f = 0,\ f_1 = \ 0\ \ldots f_5 = 0,\quad \Sigma f_{ij} du_i du_j = 0,$$

$$\Sigma f_{ijk} du_i du_j du_k + 3\,\Sigma f_{ij} d^2u_i du_j = 0,$$

$$\Sigma f_{ijkl} du_i du_j du_k du_l + 6\,\Sigma f_{ijk} du_i du_j d^2u_k + 3\,\Sigma f_{ij} d^2u_i d^2u_k + \Sigma f_{ij} du_i d^3u_j = 0.$$

We see that this S_8 contains the hyperosculating S_6 and the tangent S_5. In a similar manner it is seen that the osculating S_3's which have an osculating plane fixed generate an S_7 which is contained in the S_8 and which contains the hyperosculating S_6. If du_i defines one of the sixteen principal directions, the above S_8 becomes an S_7, etc.

We saw that a hyperosculating S_6 defined by any direction through a point x hyperosculates along a whole cone of directions passing through x. If, however, the given direction is one of the sixteen principal directions, the hyperosculating S_6 becomes indeterminate; that is,

there is an unlimited number corresponding to this direction. But an S_6 may be determined in this case which hyperosculates four points. There are sixteen such S_6's.

In ruled space of four dimensions we have :

All the linear series C_1 defined by taking a line p of V_5 and four lines consecutive to it, if only the last of the four consecutive lines is allowed to vary, are contained in a linear series C_5.

The tangent congruence to V_5 (the base of the ∞^4 linear systems C_5 tangent to V_5 in p) has three-line contact with all the ruled surfaces passing through p in one of the sixteen principal directions. There is a C_3 which has four-line contact with all the surfaces traced on V_5 tangent to a principal direction. There are sixteen such series C_3. Each one contains the tangent congruence.

A curve all of whose tangent lines lie in Φ we saw represents a developable surface in S_4 ; hence, *The lines of V_5 can be grouped into ∞^4 developable surfaces in sixteen ways ; that is, sixteen of these developables pass through each line.*[5] *Each tangent plane to one of these developables osculates V_5.* (The pencils of lines lying in the tangent plane and having their vertices on the edge of regression contain three infinitely near lines of V_5.)

There are sixteen curves of V_5 (curves enveloped by lines of V_5) *having a given line p for tangent line such that the osculating planes containing p determine the sixteen osculating pencils of which p is a part.*

7. *Other important directions.* Equations (14) set up a correspondence between two directions t and t' defined by du_i and δu_i. This correspondence is reciprocal, and we saw that the coincidences define the sixteen principal directions. This correspondence between t and t', however, is not projective, as can best be seen as follows. Consider V_5 as the intersection of four hypersurfaces in S_9. If the tangent space π_5 is cut by an arbitrary hyperplane t and t' are represented by points in this S_4, the tangent hyperplanes to the hypersurfaces at points of V_5 cut V_5 in V_4's having a conical point. The cones formed by the tangent lines at these conical points are represented in the S_4 (intersection of π_5 by a hyperplane) by quadrics. Then t' is the polar of t with respect to these four quadrics ; hence,

When t describes a line, t' describes a curve generated by the intersection of corresponding hyperplanes in S_4 of four projective pencils and therefore is a curve of order four.

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[5] This system of sixteen developables does not contain all the developables which may pass through a given line, but there are no others having the properties mentioned above.

The sixteen principal directions are represented in the S_4 by the sixteen points of intersection of the four quadrics.

The correspondence between t and t' will not be unique if t coincides with one of the vertices of the self-conjugate five-point with respect to any two of the quadrics. By choosing the defining hypersurfaces differently the self-conjugate points can be chosen in ∞^2 different ways These points are in fact the vertices of the cones belonging to the three-parameter family of quadrics defined by the four quadrics above. Now, if du_i be taken as the homogeneous coordinates of the points of the S_4 above, it is seen at once that the coordinates of the vertices of these cones are given by

$$(17) \qquad \sum_h \left(\xi, \frac{\partial^2 x}{\partial u_h \partial u_k}\right) \partial u_h = 0, \qquad k = 1, 2, 3, 4, 5.$$

(Two parameters are independent.) We shall next show that these directions are such that along them a hyperplane has two-point contact with V_5.

The condition that the hyperplane $(\xi, x) = 0$ be tangent to V_5 at x is given by

$$(18) \qquad (\xi, x) = 0, \qquad \left(\xi, \frac{\partial x}{\partial u_1}\right) = 0 \ \ldots \left(\xi, \frac{\partial x}{\partial u_5}\right) = 0.$$

If it also passes through $x + dx$, then

$$(\xi, x + dx) = (\xi, x) + (\xi, dx) = (\xi, x) + \sum \left(\xi, \frac{\partial x}{\partial u_i}\right) du_i = 0.$$

From (18) it is at once seen that this condition is verified. Now, in order that the hyperplane be tangent at $x + dx$, we must have

$$\left(\xi, \frac{\partial x}{\partial u_i} + \frac{\partial}{\partial u_i} dx\right) = \left(\xi, \frac{\partial x}{\partial u_i}\right) + \left(\xi, \sum \frac{\partial^2 x}{\partial u_i \partial u_k} du_k\right) = 0;$$

hence from (18)

$$\left(\xi, \sum \frac{\partial^2 x}{\partial u_i \partial u_k} du_k\right) = \sum \left(\xi, \frac{\partial^2 x}{\partial u_i \partial u_k}\right) du_k = 0,$$

which is equation (17) again. Hence the directions defined by (17) are those along which a hyperplane may have two points of contact with V_5. These hyperplanes are called *stationary hyperplanes.*

Through each point of V_5 pass ∞^2 curves such that along these curves hyperplanes have two-point contact with V_5.

Tangent π_5's at two consecutive points of one of these curves do not intersect in a line but in general in a plane. Then in S_4 we have the following theorem:

Through any line p of V_5 may be passed ∞^2 ruled surfaces in such a manner that there are linear series C_5 tangent to V_5 in two lines of the ruled surface infinitely near.

II. Four-Parameter Families V_4.

8. Let V_4 be the variety on Φ whose points represent the family of lines. The tangent lines to V_4 at a point x generate a linear space π_4 of four dimensions. Through π_4 pass ∞^6 S_7's and hence, *There are ∞^6 linear series C_4 tangent to a four-parameter family of lines in a line of the family.*

The osculating spaces of the curves on V_4 which pass through a given point x are determined by the points

$$f = 0, \tag{1}$$

$$\Sigma f_i du_i = 0, \tag{2}$$

$$\Sigma f_{ik} du_i du_k + \Sigma f_i d^2 u_i = 0, \tag{3}$$

$$\Sigma f_{ijk} du_i du_j du_k + 3 \Sigma f_{ik} du_i d^2 u_k + \Sigma f_i d^3 u_i = 0, \tag{4}$$

$$\Sigma f_{ijkl} du_i du_j du_k du_l + 6 \Sigma f_{ijk} du_i du_j d^2 u_k + 3 \Sigma f_{ij} d^2 u_i d^2 u_j + 4 \Sigma f_{ij}\, du_i d^3 u_j + \Sigma f_i\, d^4 u_i = 0. \tag{5}$$

By the same reasoning previously used it is at once seen that the osculating S_4's which have the same osculating S_3 in common generate an S_7 and the osculating planes having a tangent line in common generate an S_5.

The S_5 generated by the osculating planes having the tangent line (having direction du_i) in common is determined by the points

$$f = 0, \quad f_1 = 0, \quad f_2 = 0, \quad f_3 = 0, \quad f_4 = 0, \quad \Sigma f_{ik} du_i du_k = 0.$$

It is then the space which joins π_4 to the point

$$\Sigma f_{ik} du_i du_k = 0. \tag{6}$$

Now, if du_i are allowed to vary, (6) will generate a variety of three dimensions and order eight[6] V_3^8, and hence the osculating planes of

[6] See Segre, "Prelimnai di una teoria della varità luoghi di spazi," Rendiconto di Palermo, **30**, 104 (1910).

all the curves which pass through a given point x will generate a variety of order eight and dimensions eight V_8^8, which has the tangent space π_4 for vertex.

Then in S_4

All the C_1's which contain a fixed line p of V_4 and two fixed lines infinitely near to it are contained in a C_4.

There is a C_2 which contains all the lines infinitely near, to infinitesimals of the second order, to a given line of V_4 in a given direction (direction being defined by two lines infinitely near of the first order.)

If the directions are varied, the C_2's above will generate a five-parameter family of lines of order eight which has the lines of a ruled surface C_1 for double lines.

9. *Noteworthy directions on V_4.* Through each point of V_4 pass ∞^2 important directions. In these directions there are curves whose osculating plane has four consecutive points in common with the V_4. If this is the case, the points (1), (2), (3), (4) must lie in the same plane. If du_i and d_2u_i are held fixed for the moment and the d^3u_i are allowed to vary, it is at once seen that if these four points lie in a plane then it is necessary that the points

$$(1),\ (2),\ (3),\ \Sigma f_{ijk}\, du_i du_j du_k + 3\,\Sigma f_{ik} du_i d^2u_k = 0,$$

$$\Sigma f_i d^3u_i = 0,$$

lie in the same S_3, and consequently the points

$$(7) \qquad f = 0,\quad f_1 = 0,\quad f_2 = 0, f_3 = 0,\quad f_4 = 0,$$

$$(6) \qquad \Sigma f_{ik} du_i du_k = 0,$$

$$(8) \qquad \Sigma f_{ijk}\, du_i du_j du_k + 3\,\Sigma f_{ik} du_i d^2u_k = 0,$$

will lie in an S_5. Conversely it is easily shown that if these points lie in an S_5 then values of d^3u_i can be determined so that the original four points (1), (2), (3), (4) will lie in a plane. Now, writing the last term of (8) in the form

$$d^2u_1 \Sigma f_{1k} du_k + d^2u_2 \Sigma f_{2k} du_k + d^2u_3 \Sigma f_{3k} du_k + d^2u_4 \Sigma f_{4k} du_k,$$

and noting that (6) can be written in a similar form where the d^2u_i are replaced by du_i we conclude that if the points (6), (7), (8) lie in an S_5 it is necessary that the ten points

$$(7) \qquad f = 0,\quad f_1 = 0,\quad f_2 = 0,\quad f_3 = 0,\quad f_4 = 0,$$

$$(9) \qquad \Sigma f_{ik} du_i = 0, \qquad k = 1, 2, 3, 4,$$

$$(10) \qquad \Sigma f_{ijk} du_i du_j du_k = 0,$$

will lie in an S_3, and conversely, if these ten points lie in an S_3, d^2u_i and d^3u_i can be so determined that the original four points will lie in a plane. The condition that these ten points lie in a plane is

$$(11) \qquad \left| x \; \frac{\partial x}{\partial u_1} \cdots \frac{\partial x}{\partial u_4} \; \Sigma f_{ik} du_i du_k \cdots \Sigma f_{ijk} du_i du_j du_k \right| = 0.$$

Hence, if the du_i be considered as the homogeneous coordinates of the directions of the tangent lines at a point of V_4, we see that *Through each point of V_4 pass ∞^2 directions whose tangent lines form a cone of order seven such that along these directions there are curves whose osculating planes cut V_4 in four consecutive points.*

Then in S_4 we have

Through each line p of a four-parameter family V_4 pass ∞^2 directions such that there is a ruled surface tangent to each of these directions (containing the two infinitely near lines which determine the direction) *and possessing the property that four infinitely near generators are cut by one line.*

It is known that four lines determine a C_0 (the intersection of six C_5's), but through four infinitely near lines of one of these ruled surfaces pass a whole pencil of C_0's. There are C_1's which have five lines infinitely near in common with V_4.

A four-parameter family of lines can be generated by ∞^3 ruled surfaces such that four consecutive generators are cut by a line. This can be done in ∞^2 different ways.

10. Another set of *important* directions are those along which a tangent C_4 to V_4 has double contact. In S_9 a tangent S_7 to a V_4 is defined by the equations

$$(12) \qquad \begin{aligned} &(\xi, x) = 0, \quad \left(\xi, \frac{\partial x}{\partial u_1}\right) = 0 \cdots \left(\xi, \frac{\partial x}{\partial u_5}\right) = 0, \\ &(\xi, x) = 0, \quad \left(\xi, \frac{\partial x}{\partial u_1}\right) = 0 \cdots \left(\xi', \frac{\partial x}{\partial u_5}\right) = 0. \end{aligned}$$

If this S_7 passes through $x + dx$,

$$(\xi, x + dx) = 0, \qquad (\xi', x + dx) = 0.$$

Expressing dx in terms of du_i and applying (12), it is at once evident that these equations are satisfied. Now, if S_7 is also tangent at this point,

$$\left(\xi, \frac{\partial}{\partial u_i}(x + dx)\right) = 0, \qquad i = 1, \ldots 4.$$

$$\left(\xi', \frac{\partial}{\partial u_i}(x + dx)\right) = 0, \qquad i = 1, \ldots 4.$$

Expanding these equations and making use of (12), this condition reduces to

$$\sum\left(\xi, \frac{\partial^2 x}{\partial u_i \partial u_k}\right) du_k = 0, \qquad i = 1, \ldots 4.$$

(13)

$$\sum\left(\xi', \frac{\partial^2 x}{\partial u_i \partial u_k}\right) du_k = 0, \qquad i = 1, \ldots 4.$$

For these equations to be compatible requires five conditions, but as we saw a tangent S_7 depends upon six constants (six ξ's are arbitrary); hence there are ∞^1 solutions of (13), that is, there are ∞^1 S_7's which have two-point contact with V_4.

Then in S_4 *There are* ∞^1 C_4*'s tangent to* V_4 *in a line* p *which are also tangent to* V_4 *in a line infinitely near to* p.

The consecutive tangent π_4's to V_4 (in S_9) along one of the directions above have a line in common. This line of intersection and the line joining the point of contact are in a sense conjugate directions.

III. Three-Parameter Families V_3.

11. The tangent space π_3 to V_3 at a point x is of three dimensions; hence there are ∞^9 S_6's passing through π_3 (and consequently tangent to V_3). Then in S_4 *There are* ∞^9 C_3*'s tangent to a three-parameter family of lines* V_3 *in a given line.*

The osculating S_4's which have a fixed osculating S_3 in common generate an S_6. Then in S_4 *The ruled surfaces* C_1, *which have in common four infinitely near fixed lines of* V_3 *generate a* C_3.

12. The osculating planes which pass through a given direction generate an S_4, and if the direction is varied these S_4's will generate a cone V_6^4 having π_3 for vertex. The S_4 is determined by the points

$$f = 0, \quad f_1 = 0, \quad f_2 = 0, \quad f_3 = 0, \qquad \Sigma f_{ik} du_i du_k = 0.$$

Then if du_i is varied (that is, if the tangent line is varied) the cone is generated by the S_4's which project π_3 from the points of the surface generated by

$$\Sigma f_{ik} du_i du_k = 0,$$

and hence is a cone of order four and dimensions six. Then

In a three-parameter family of lines all the lines infinitely near a given line to infinitesimals of the second order lie in a three-parameter family of order four which has the lines represented by the intersection of π_3 with Φ for double lines.

In general there are no osculating planes of V_3 which hyperosculate. Hence in S_4 four consecutive lines of a three-parameter family are not in general cut by a fifth line.

13. *Special directions.* There are ∞^1 directions such that hyperplanes can be found which are tangent to V_3 in three consecutive points. Let the three points be

$$x, \qquad x' = x + dx, \qquad x'' = x + 2\,dx + d^2x.$$

If the hyperplane $(\xi, x) = 0$ is tangent at x,

$$(1) \quad (\xi, x) = 0, \quad \left(\xi, \frac{\partial x}{\partial u_1}\right) = 0, \quad \left(\xi, \frac{\partial x}{\partial u_2}\right) = 0, \quad \left(\xi, \frac{\partial x}{\partial u_3}\right) = 0,$$

and if it passes through x',

$$(\xi, x + dx) = 0,$$

which from (1) is seen to be true. Now, if (ξ, x) is tangent at x',

$$\left(\xi, \frac{\partial(x + dx)}{\partial u_i}\right) = 0,$$

which on expanding and applying (1) reduces to

$$\sum\left(\xi, \frac{\partial^2 x}{\partial u_i \partial u_k}\right) du_k, \qquad i = 1, 2, 3.$$

If the hyperplane passes through x'', it is at once seen that

$$\sum\left(\sum\left(\xi, \frac{\partial^2 x}{\partial u_i \partial u_k}\right) du_i\right) du_k = 0,$$

which is seen to be satisfied if (2) is satisfied; that is, if the hyperplane is tangent at x', it passes through x''. If it is also tangent at x'',

$$\left(\xi, \frac{\partial}{\partial u_i}(x + 2\,dx + d^2x)\right) = 0.$$

Expanding and using the previous relations, this reduces to

$$\left(\xi, \sum_{ij} \frac{\partial^2 x}{\partial u_i \partial u_j \partial u_k} du_i du_j\right) + 3\left(\xi, \sum \frac{\partial^2 x}{\partial u_i \partial u_j} du_i d^2 u_j\right) = 0.$$

The second term of this expression vanishes if (2) is satisfied; hence

$$\left(\xi, \sum_{ij} \frac{\partial^2 x}{\partial u_i \partial u_j \partial u_k} du_i du_j\right) = 0, \qquad k = 1, 2, 3.$$

The hyperplane (ξ, x) will then have three points of contact infinitely near if the ten equations (1), (2), (3) are satisfied. This requires one relation, and hence there are ∞' directions du_i of V_3 generating a cone of order nine along which a hyperplane has three infinitely near points of contact with V_3. Then

In a three-parameter family of lines through each point pass ∞^1 *directions such that a* C_5 *will be tangent in three lines of the system infinitely near.*

IV. Application to the Geometry of the Circle.

14. If the equation of the sphere is written in the form

$$t\,(x^2 + y^2 + z^2) + 2\,(ax + by + cz) + 2\,d = 0,$$

the quantities a, b, c, d, t may be taken as the coordinates of the sphere. The spheres of the pencil determined by two spheres (a, b, c, d, t), (a', b', c', d', t') will have coordinates

$$(a + \lambda a', b + \lambda b', c + \lambda c', d + \lambda d', t + \lambda t').$$

Then, if we consider the circle defined by this pencil as representing the pencil, we see that the circle in ordinary space corresponds to the line in S_4 joining the points

$$(a, b, c, d, t), (a', b', c', d', t').$$

The coordinates of the circle can then be taken as the two-row determinants of the matrix

$$\left\| \begin{matrix} a & b & c & d & t \\ a' & b' & c' & d' & t' \end{matrix} \right\|.$$

These coordinates are identical with those used of the line in S_4.

The lines in S_4 which lie in an S_3 correspond to the circles which cut a given sphere orthogonally.[7]

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[7] For the explanation of this and the following statements the reader is referred to the paper of the author previously referred to.

Two lines in S_4 which intersect correspond to circles which lie on the same sphere.

A pencil of lines in S_4 corresponds to a pencil of circles on a sphere.

All the lines in S_4 which pass through a given point correspond to all the circles on a given sphere.

The lines in S_4 which lie in a plane correspond to the circles which pass through two fixed points.

Developable surfaces in S_4 correspond to annular surfaces and ruled surfaces to circled surfaces.

We are now able to interpret the preceding theorems in S_4 in circles.

15. FIVE-FOLD INFINITE SYSTEMS OF CIRCLES U_5. *In each circle C of U_5 there are ∞^4 linear systems A_5 tangent to it. Each tangent A_5 cuts U_5 in a circled surface which has C for multiple generator of order* 16.

The circles of U_5 can be grouped into ∞^4 annular surfaces such that the osculating pencil of circles (the pencil determined by C and the two points in which it cuts the curve corresponding to the cuspidal edge of a developable) [8] *has three circles infinitely near in common with U_5.* This can be done in sixteen different ways.

Through each circle C of U_5 pass sixteen annular surfaces having the pencil of spheres determined by C for tangent pencil such that the osculating congruence determined by three infinitely near circles of the annular surface determine the sixteen osculating pencils of which C is a part.

Through each circle C of U_5 pass ∞^2 circled surfaces such that there are linear series A_5 tangent to U_5 in two consecutive circles of such surfaces.

16. FOUR-PARAMETER FAMILIES U_4. *There are ∞^6 linear series A_4 tangent to U_4 in a given circle C.*

All the linear circled surfaces [9] *A_1 which contain C and two circles of U_4 infinitely near to it generate a linear series C_4.*

All the circles of U_4 infinitely near C to infinitesimals of second order, in a given direction (defined by two consecutive circles) *lie in a linear series A_2. If the direction is varied, the linear series will generate a U_5 having all the circles of a circled surface for double elements.*

Through each circle C of U_4 pass ∞^2 directions such that there are circled surfaces tangent to each direction (the circled surfaces contain

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[8] See "Circles orthogonal to a given sphere," Annals of Mathematics, series 2, **8**, 57.

[9] The term linear circled surface is used to indicate the circled surface common to five-linear circle complexes. Linear does not apply to the order or class of the surface.

the two infinitely near circles which define the direction) *such that four infinitely near generators are cut in two points by one circle.*

The system U_4 can be generated in ∞^2 ways by ∞^3 circled surfaces having the above property.

There are ∞^2 linear series A_4 tangent to U_4 in a circle C and a circle infinitely near to C.

17. THREE-PARAMETER FAMILIES U_3. *There are ∞^9 linear series A_3 tangent to U_3 in a given circle C.*

The linear circled surfaces which have C and three consecutive circles to C in common generate a linear series A_3.

The circles infinitely near to the first and second order to a given article C of U_3 lie in a three-parameter family of order four having five circles of a penticycle for double elements.

There are ∞^1 directions such that a linear system A_5 will have three-circle contact with U_3.

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